(Near) Optimal Adaptivity Gaps for Stochastic Multi-Value Probing

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September 21, 2019

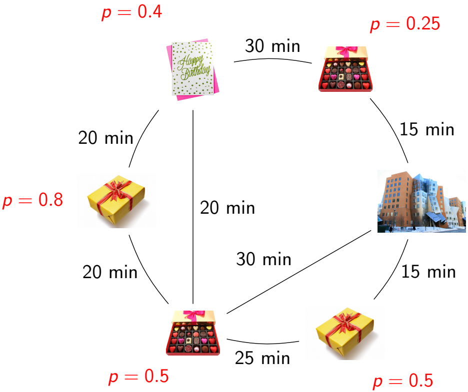
Domagoj Bradac

Sahil Singla

[2]Slides are based on a deck by Sahil Singla.

Outline

Motivating example: Birthday party



- Only 1 hour before deadline!

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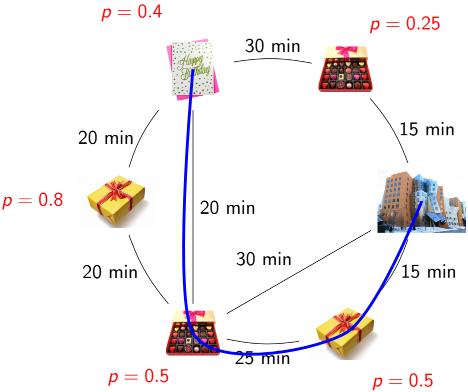
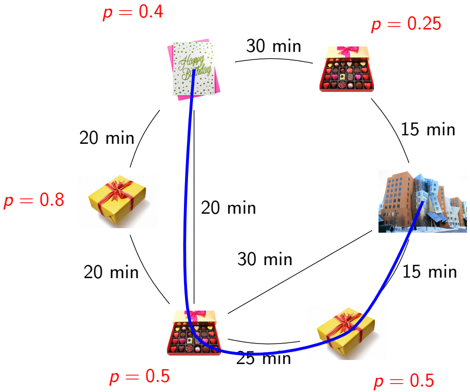


- Only 1 hour before deadline!
- Remark: if probabilities 1 & nodes distinct, then **Orienteering Problem** [Blum et al. FOCS'03].

- Probabilities of being active: independent.



Things to note

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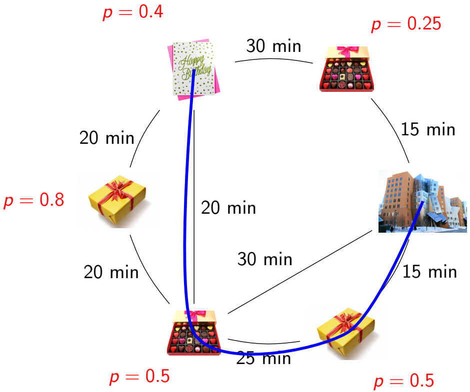

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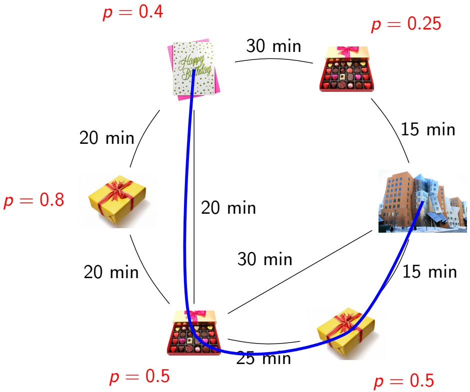

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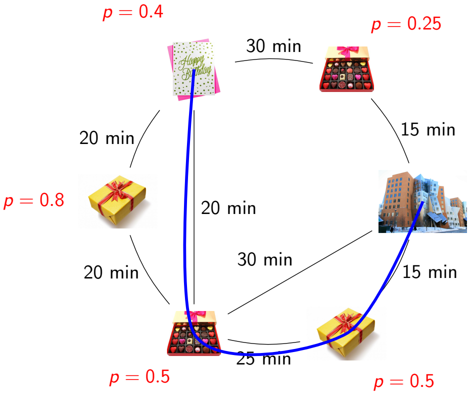
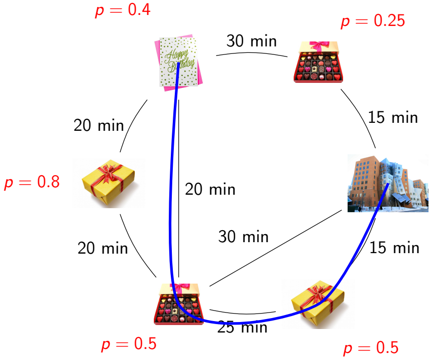
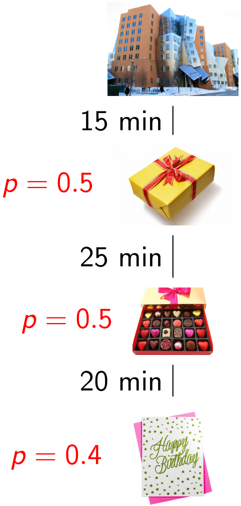
- Probabilities of being active: independent.
- Objective: # distinct items.
- Constraint: 1 hour in the given metric.
- Goal: maximize the expected objective value.

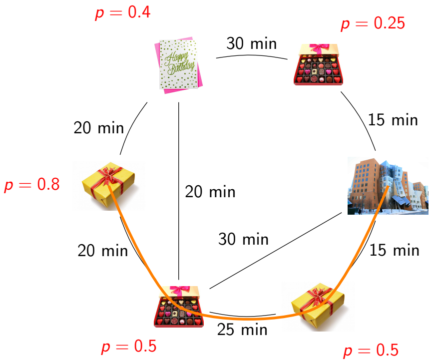




$$0.5 + 0.5 + 0.4 = 1.4$$

Can we do better?





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$$0.5(1 + 0.5 + 0) + 0.5(0 + 0.5 + 0.8) = 1.4$$

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Problem definition: Stochastic probing

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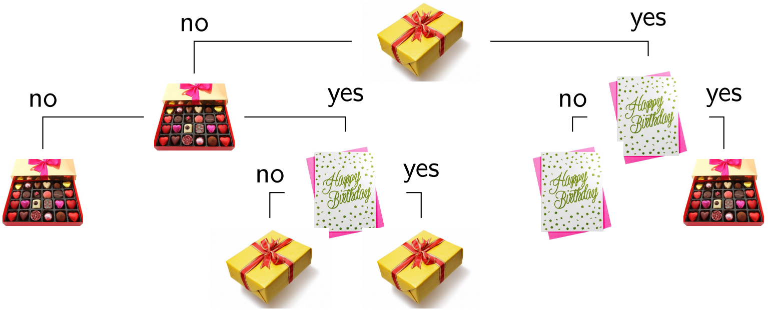
- Goal: Maximize $f(\mathrm{Probed} \cap A)$.
  - i.e., $\mathbb{E}_A[f(\mathrm{Probed} \cap A)]$

Adaptive vs. non-adaptive strategies

Adaptive: a decision tree where every root-leaf path is feasible.



Non-adaptive:

Outline

Adaptivity gap

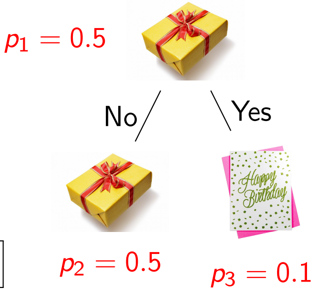
Definition (Adaptivity gap)

Ratio of the best adaptive to best non-adaptive strategy.

$$\text{AdaptivityGap} := \frac{\mathbb{E}[\text{Adap}]}{\mathbb{E}[\text{NA}]}$$

Adaptive example: probe 1 $\rightarrow$ probe 2 OR 3.

$$\begin{aligned}\mathbb{E}[\text{Adap}] &= 0.5(1 + 0.1) + 0.5(0 + 0.5) \\ &= 0.8 \\ \mathbb{E}[\text{NA}] &= 1 - 0.5 \cdot 0.5 \\ &= 0.75\end{aligned}$$

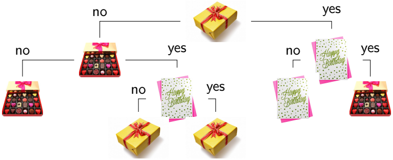


Main question: How large can the gap be?

Why care about adaptivity gaps?

Adaptive strategy concerns:

- Can be exponentially sized.
- How to compute?
- How to represent?



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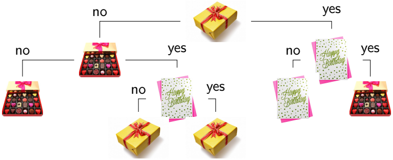

Best non-adaptive: select a feasible $\mathrm{Probed} \in \mathcal{C}$ in the beginning to

$$\max \mathbb{E}_{A\sim p}[f(\mathrm{Probed} \cap A)].$$

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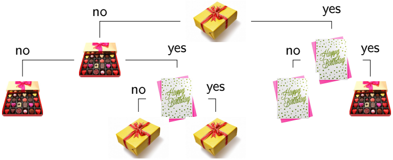

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- Easier to find: $g(S) = \mathbb{E}_{A\sim p}[f(S \cap A)]$ is often submodular.

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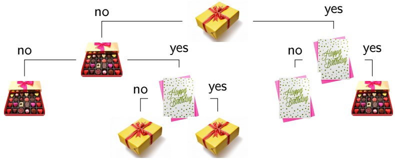

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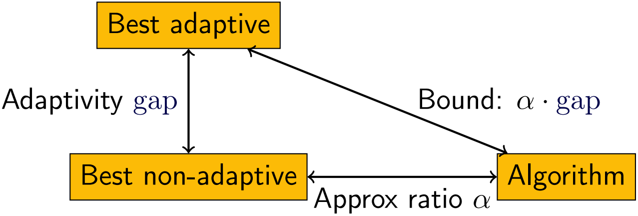
Concern: large adaptivity $\mathrm{gap} := \frac{\mathbb{E}[\mathrm{Adap}]}{\mathbb{E}[\mathrm{NA}]}$.

Small adaptivity gap

- Assume α is small.



Question: For what constraints and functions is the gap small?

Outline

Our results

Theorem

Adaptivity gap is 2.

- *Always* $\text{gap} \leq 2$, *there exists an example where gap* $= 2$.
- *Function* $=$ *monotone submodular.*
- *Constraints* $=$ *downward closed.*

- Downward closed: If a set can be probed then also its subsets can.
- Submodular: E.g., # of distinct elements.

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Theorem

Adaptivity gap is between $\Omega(\sqrt{k})$ *and* $O(k \log k)$.

- *Function* $=$ *weighted rank of k-matroid intersection.*
- *Constraints* $=$ *downward closed.*

Prior work

Reference	Function	Constraints	Gap LB	Gap UB
[GN'13]	k_1-matroid intersection	k_2-matroid intersection		$O\left((k_1+k_2)^2\right)$
[GNS'16]	k-matroid intersection	downward closed		$O(k^4 \log kn)$
[an'16]	monotone submodular	1-matroid	$\frac{e}{e-1} \approx 1.58$	$\frac{e}{e-1} \approx 1.58$
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Upper bound: Proof Steps

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Adaptivity gap is at most 2.

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Steps:

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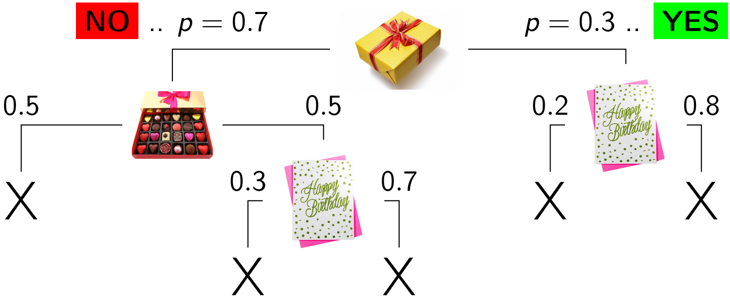
Steps:

1. Transform an adaptive strategy to a non-adaptive.
2. Prove $\mathbb{E}[NA] \geq \frac{1}{2}\mathbb{E}[Adap]$.

Upper bound: Two ideas

(1) Take a random root-leaf path

- Only show existence



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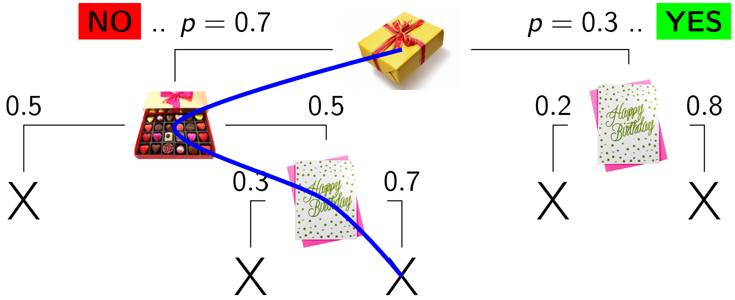

Blue path prob $= 0.7 \cdot 0.5 \cdot 0.7$

- Here *Adap* gets 2.
- Here *NA* gets $0.3 + 0.5 + 0.7 = 1.5$
- Goal to show: $\mathbb{E}[NA] = \mathbb{E}[RandomPath] \geq \frac{1}{2}\mathbb{E}[Adap]$.

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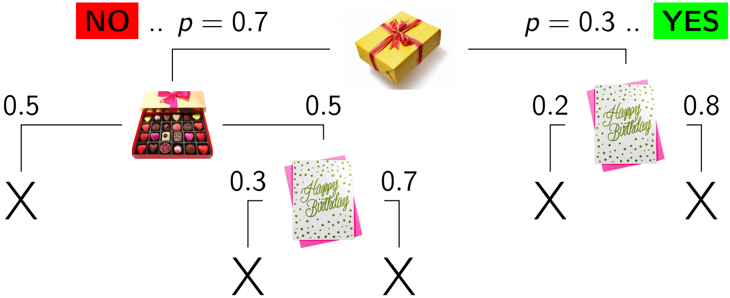

(2) Node-by-node induction

- Convert *NA* to a "greedy" algorithm for induction.

Upper bound: Proof

Goal: $\mathbb{E}[RandomPath] \geq \frac{1}{2}\mathbb{E}[Adap]$.

Def: $I =$ is the root active in *Adap*?

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$$Adap \quad = \quad \mathbb{E}_I[f(I) + Adap(T_I, f_I)]$$

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Non-adaptive:

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Induction hypothesis:

$$\mathbb{E}_{I,R}[Adap(T_i, f_{I \cup R})] \leq 2 \cdot \mathbb{E}_{I,R}[NA(T_I, f_{I \cup R})]$$

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Conclusion

Generalizations of stochastic probing:

- Multi-value setting.
- k-extendible systems.
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