Universally-Optimal $(1+\varepsilon)$-Approximate Shortest Path and Transshipment in the Distributed Setting

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21 Oct 2021

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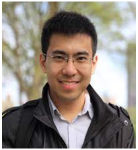

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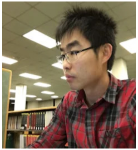

Mingquan Ye

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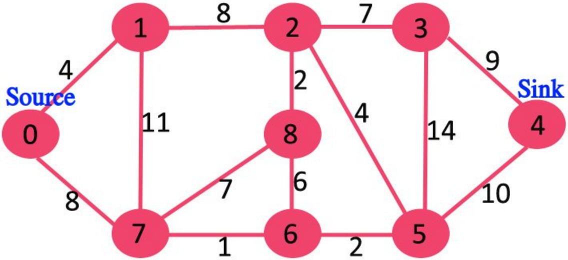
- Given a n-vertex undirected graph where edges have weights in the set $\{1, 2, \ldots, n^{O(1)}\}$. Compute shortest path from source to all other nodes.



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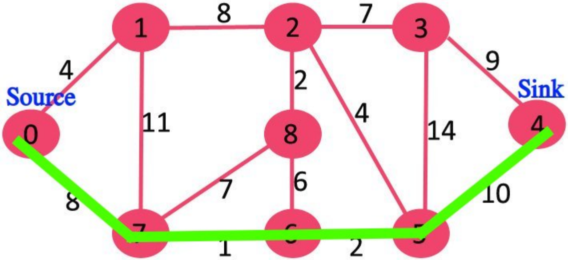

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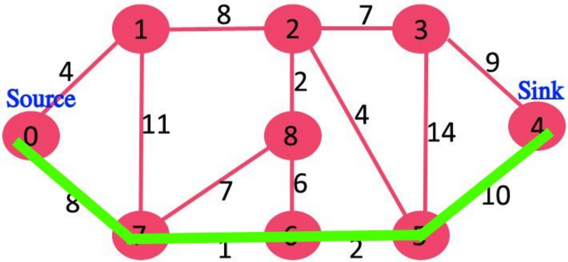

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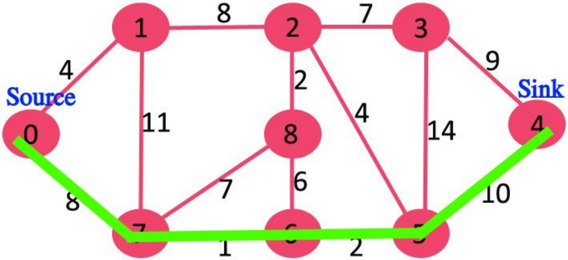

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What about parallel or distributed settings? The problem seems much harder, processing long paths that fork and merge seems inherently sequential at first glance.

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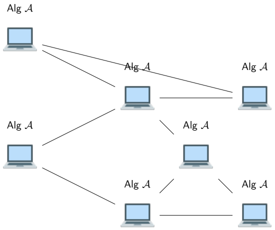

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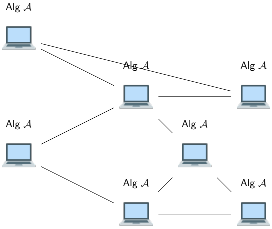

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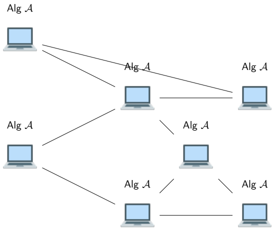

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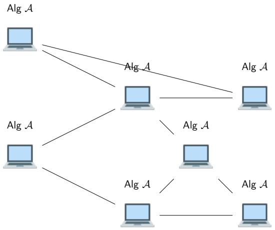

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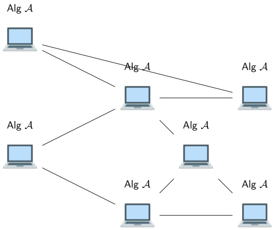

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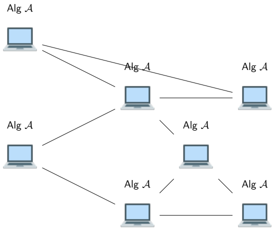

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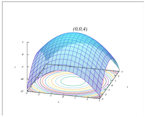
Recent results that develop fastest parallel SSSP algorithm use techniques from **continuous optimization**.



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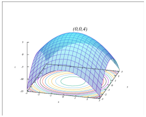

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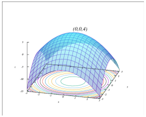

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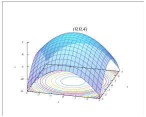

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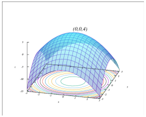

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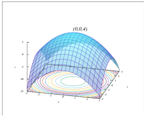

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Thm: $OPT(G) =_{\tilde{\Theta}(1)} ShortcutQuality(G)$ [Haeupler, Wajc, Zuzic; 2021]

1 Introduction

2 Main Ideas

- Idea 1: Transshipment generalizes shortest path
- Idea 2: Transshipment boosting
- Idea 3: Approximately Solving Transshipment
- Idea 4: Distributed Implementation

3 Conclusion

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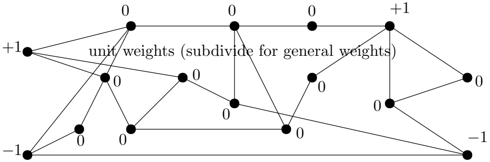

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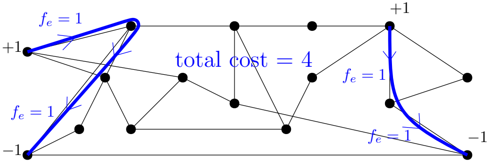

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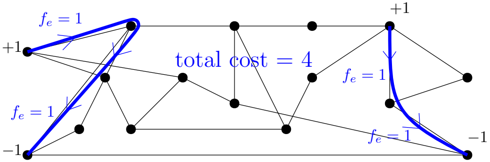

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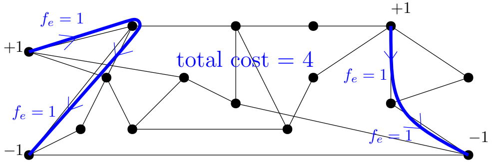

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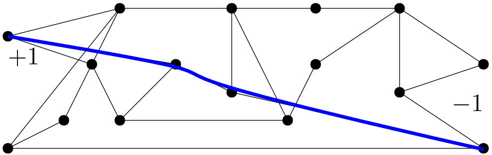
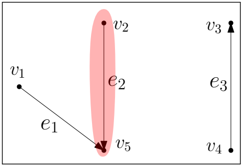
Note. Generalizes $(s - t)$ shortest path. (Also generalizes SSSP.)



Transshipment: a primal-dual formulation

Write the graph $G = (V, E)$ using the node-edge incidence matrix B.
Note: we orient edges arbitrarily.



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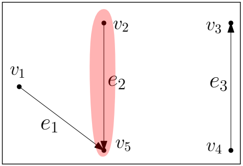

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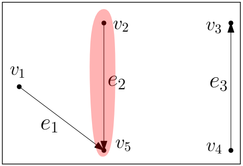

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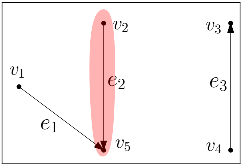

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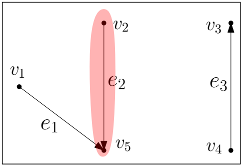

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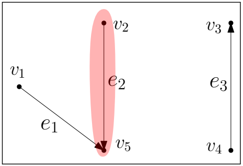

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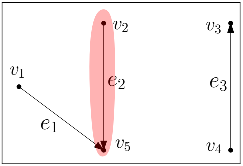

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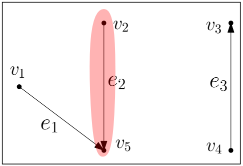

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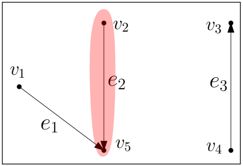

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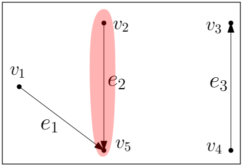

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Theorem ([Sherman; 2013], [BFKL; 2016], [Zuzic; unpublished])

Fix G. Suppose we are given an oracle $O_G(\cdot)$ which, given a demand d, outputs an α-approximate feasible dual $O_G(d)$. There is an algorithm that produces a $(1+\varepsilon)$-approximate feasible dual by calling $O_G(\cdot)$ at most $\mathrm{poly}(\alpha, \varepsilon^{-1}, \log n)$ times.

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Corollary

Given such a (dual) $n^{o(1)}$-approximation oracle, we can solve $(1+\frac{1}{n^{o(1)}})$-approximate transshipment in $n^{o(1)}$ oracle calls.

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Goal: find an approximate ~~dual~~ solution

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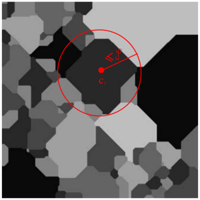
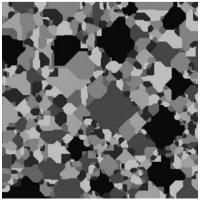
1. For each i, the center c_i is within distance ρ of every other node in the induced subgraph $G[S_i]$, w.h.p.
2. For all $x, y \in V$, the probability they are in different clusters is at most $2^{\sqrt{\log n}} \cdot \frac{dist_G(u,v)}{\rho}$.





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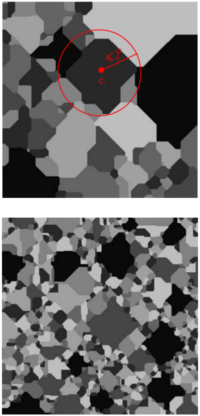
Theorem (Prior work [Haeupler, Li; 2018])

LDDs can be sampled in $OPT(G)n^{o(1)}$ CONGEST rounds.



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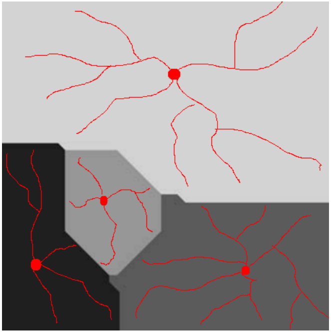

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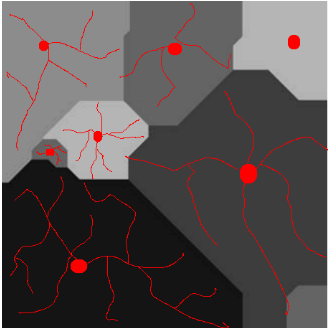

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Algorithm 0: Oblivious routing for TS.

1. Let $\rho := 2^{(\log n)^{3/4}}$ (LDD radius).
2. For $i = 1, 2, \ldots, (\log n)^{1/4}$ repeat the following:
   1. For $j = 1, 2, \ldots, g := 2^{(\log n)^{3/4}}$.
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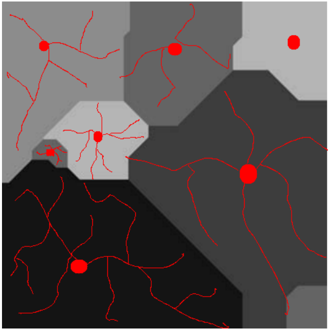
   2. Update the demand to reflect the transport.
3. Route all remaining demand to a common node along any spanning tree.



When $i = (\log n)^{1/4}$, radius is $\rho^i = \text{poly}(n)$ and LDD has a single cluster.

Analysis intuition.

Question: how does OPT change between steps?

Fix u, v at distance ℓ. Suppose at some step $d(u) = +1, d(v) = -1, d(\text{all else}) = 0$. Clearly, $\mathrm{OPT}_{before} = \ell$. Suppose we sampled an LDD of radius ρ. How does OPT change?

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In both cases, OPT increases by at most $2^{\sqrt{\log n}}$ factor. Hence after $(\log n)^{1/4}$ steps, it only increases by

$$\left(2^{\sqrt{\log n}}\right)^{(\log n)^{1/4}} = 2^{(\log n)^{3/4}} = n^{o(1)}$$

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- Reason 2: the transport cost in this single step would be too high.

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Corollary

Transshipment can be solved in $n^{o(1)}$ Minor-Aggregation rounds. Hence it can be implemented in $OPT(G) \cdot n^{o(1)}$ CONGEST rounds.

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