A Simple Boosting Framework for Transshipment

Goran Zuzic

ETH Zürich

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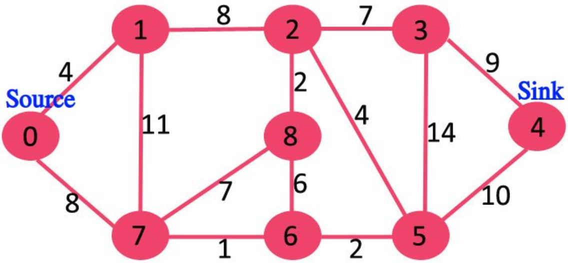
Original motivation

We want to solve the **single-source shortest path** problem (**SSSP**).

- Given an undirected graph where edges have weights. Compute shortest path from source to all other nodes.



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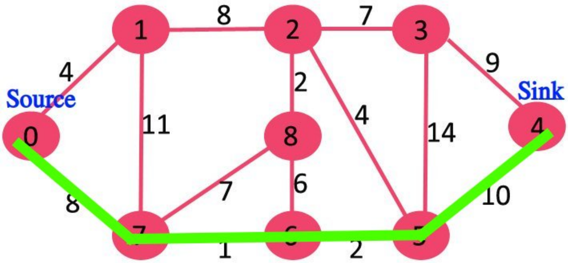

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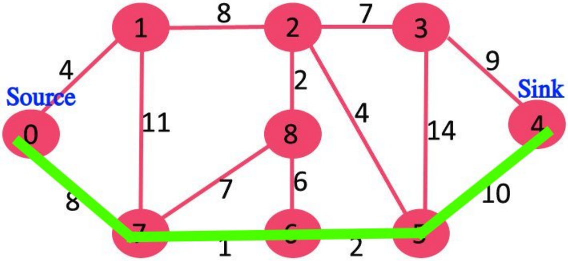

One of the oldest problems in computer science.
Sequential setting .. easy! Dijkstra's famous $\tilde{O}(m+n)$-time algorithm is optimal (modulo log).

What about parallel or distributed settings? The problem seems much harder: Dijkstra fails miserably.

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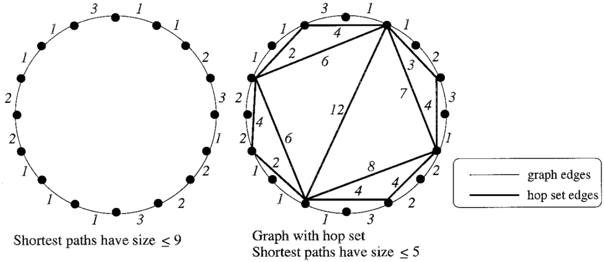
- **Hopset:** Add a small number of edges to a graph such that original shortest paths are $(1+\varepsilon)$-approximated with new paths with small number of hops. (Figure taken from Cohen'00)



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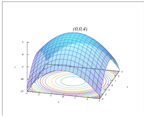
Main ideas:

- **Continuous optimization:** (Today) Generalize the shortest path to transshipment. Find an bad approximate solution and boost it to an $(1+\varepsilon)$-approximate.

Recent wins of the continious-optimization approach



Recent results based on **continuous optimization**.

Parallel.

$(1+\varepsilon)$-apx with $\tilde{O}(1)$ depth and $\tilde{O}(m)$ work. [Li; 2020] [ASZ; 2020]

$(1+\varepsilon)$-apx <u>deterministic</u> with $\tilde{O}(1)$ depth and $\tilde{O}(m)$ work. [RGHZL; 2022]

Distributed.

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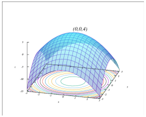

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$(1+\varepsilon)$-apx in $OPT(G) \cdot n^{o(1)}$ rounds. [ZGYHS'22].

1 Introduction

2 Main Aspects of the Solution

- Idea 1: Transshipment generalizes shortest path
- Notation: Consider the LP primal-dual formulation
- Idea 2: Transshipment boosting with duals
- Idea 3: Multiplicative weights
- Idea 4: Self-reduction of transshipment and consequences

3 Conclusion

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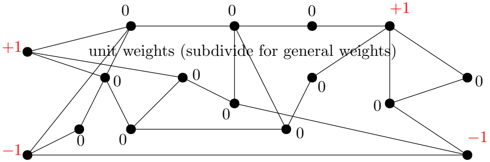

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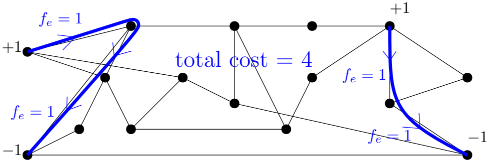

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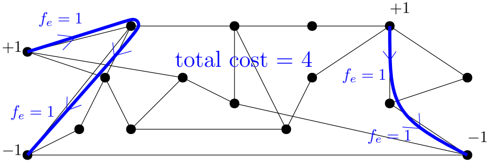

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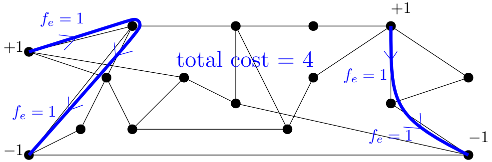

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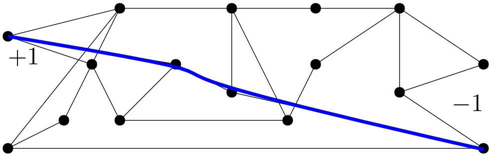
Note. Generalizes $(s - t)$ shortest path. (Also generalizes SSSP.)



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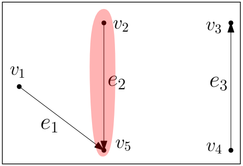
Write the graph $G = (V, E)$ using the node-edge incidence matrix B.
Steps: (1) Orient edges arbitrarily. (2) For each arc, add column to B.



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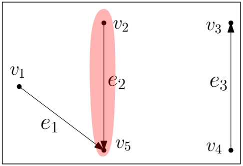

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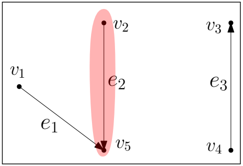

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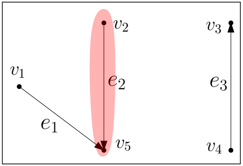

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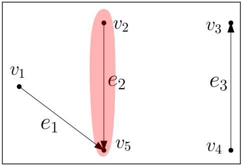

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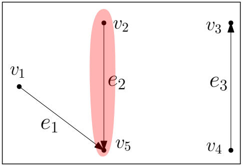

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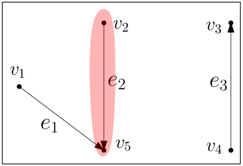

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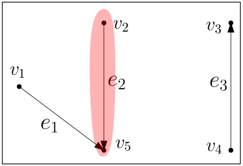

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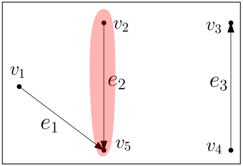

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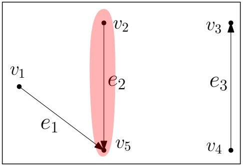

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Theorem ([Sherman; 2013], [BFKL; 2016], [Zuzic; unpublished])

Fix G. Suppose we are given an oracle $O_G(\cdot)$ which, given a demand d, outputs an α-approximate feasible dual $O_G(d)$. There is an algorithm that produces a $(1+\varepsilon)$-approximate feasible dual by calling $O_G(\cdot)$ at most $\mathrm{poly}(\alpha, \varepsilon^{-1}, \log n)$ times.

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Corollary

Given such a (dual) $n^{o(1)}$-approximation oracle, we can solve $(1 + \frac{1}{n^{o(1)}})$-approximate transshipment in $n^{o(1)}$ oracle calls.

Feasibility task

$\exists\phi \quad ||A\phi||_\infty + \langle b, \phi\rangle \leq \gamma$

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Multiplicative Weights

maintain sol. $\phi_1, \phi_2, \ldots$

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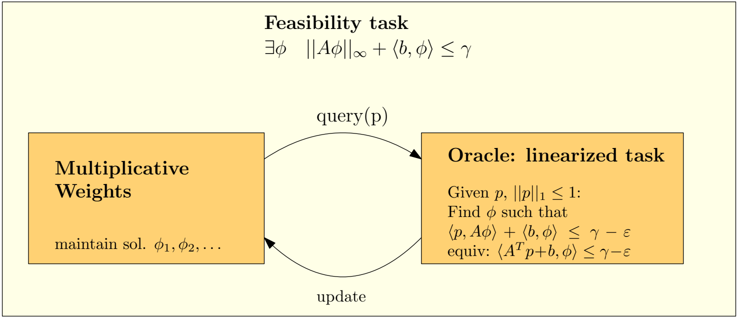

Question: How many oracle calls?

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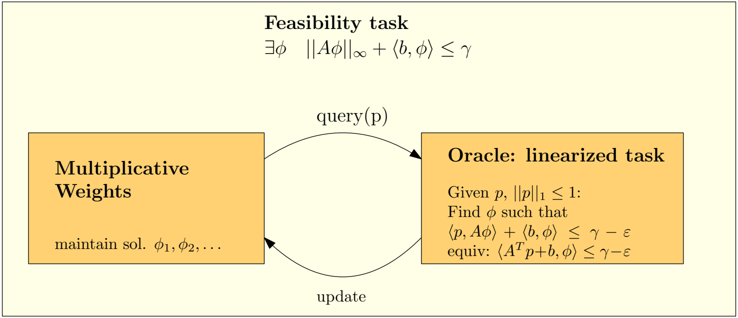

Question: How many oracle calls?
The oracle will be queried poly$(\varepsilon^{-1}, \log n, \rho)$ times.
Here, $\rho \geq \|A\phi\|_\infty$ called **width of the oracle**.

Idea 4: Self-reduction of transshipment and consequences

Dual: $\max_{\phi} \quad \langle d, \phi \rangle$ such that $\left\| B^{\top} \phi \right\|_{\infty} \leq 1$

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Multiplicative weights: $\exists \phi, \quad \frac{1}{g} \langle d, \phi \rangle \geq \left\langle p, B^\top \phi \right\rangle + \varepsilon$

Idea 4: Self-reduction of transshipment and consequences

Dual: $\max_\phi \quad \langle d, \phi \rangle$ such that $\left\| B^\top \phi \right\|_\infty \leq 1$

Binary search g: $\exists \phi, \quad \langle d, \phi \rangle \geq g, \quad \left\| B^\top \phi \right\|_\infty \leq 1$

Rewrite: $\exists \phi, \quad \frac{1}{g} \langle d, \phi \rangle \geq 1 \geq \left\| B^\top \phi \right\|_\infty$

Eliminate middle: $\exists \phi, \quad \frac{1}{g} \langle d, \phi \rangle \geq \left\| B^\top \phi \right\|_\infty$

Multiplicative weights: $\exists \phi, \quad \frac{1}{g} \langle d, \phi \rangle \geq \left\langle p, B^\top \phi \right\rangle + \varepsilon$

Rewrite: $\exists \phi, \quad \left\langle \underbrace{\frac{1}{g} d - Bp}_{d_{\text{residual}}}, \phi \right\rangle \geq \varepsilon$

Dual: $\max_\phi \quad \langle d, \phi \rangle$ such that $\left\| B^\top \phi \right\|_\infty \le 1$

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Oracle: $\exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \ge \varepsilon$. Note: oracle width $\rho \ge \left\| B^T \phi \right\|_\infty$

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Equiv: $\exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon / \rho, \quad \left\| B^T \phi \right\|_\infty \leq 1$

Self-reduction consequences

$$\textbf{Oracle: } \exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon/\rho, \quad \left\| B^T \phi \right\|_\infty \leq 1$$

Notes:

- Q: Why does this make progress at all?

Self-reduction consequences

Oracle: $\exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon / \rho, \quad \left\| B^T \phi \right\|_\infty \leq 1$

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- Q: Why does this make progress at all? Ans: Note that we can increase ρ, the final answer is still $(1+\varepsilon)$-approximate but runtime increases.

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Approximation $\rightarrow$ runtime.

$$\exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon / (\alpha \cdot \rho), \quad \left\| B^T \phi \right\|_\infty \leq 1$$

Self-reduction consequences

Oracle: $\exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon/\rho, \quad \left\| B^T \phi \right\|_\infty \leq 1$

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- Q: What if there is no solution to the new problem?

Self-reduction consequences

Oracle: $\exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon / \rho, \quad \left\| B^T \phi \right\|_\infty \leq 1$

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- Q: What if there is no solution to the new problem? Ans: Look at primal and prove the original problem has no solution.

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- Q: What if there is no solution to the new problem? Ans: Look at primal and prove the original problem has no solution.
- Computationally, we start with the dual LP and end with a dual LP. Result: Any α-approximate dual solver can be boosted to $(1+\varepsilon)$-approximate dual solver. Runtime: poly$(\alpha, \varepsilon^{-1}, \log n)$ query calls.

1. Introduction
2. Main Aspects of the Solution
   - Idea 1: Transshipment generalizes shortest path
   - Notation: Consider the LP primal-dual formulation
   - Idea 2: Transshipment boosting with duals
   - Idea 3: Multiplicative weights
   - Idea 4: Self-reduction of transshipment and consequences
3. Conclusion

Thank you!