Universally-Optimal $(1+\varepsilon)$-Approximate Shortest Path and Transshipment in the Distributed Setting

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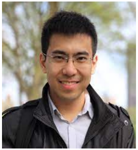
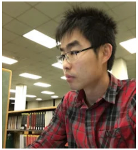
21 Oct 2021

Gramoz Goranci

Bernhard Haeupler



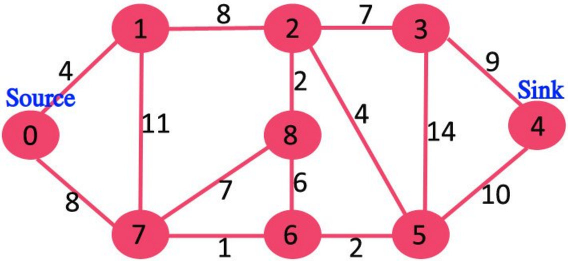
Xiaorui Sun



Mingquan Ye

We want to solve the **single-source shortest path** problem (**SSSP**).

- Given a n-vertex undirected graph where edges have weights in the set $\{1, 2, \ldots, n^{O(1)}\}$. Compute shortest path from source to all other nodes.



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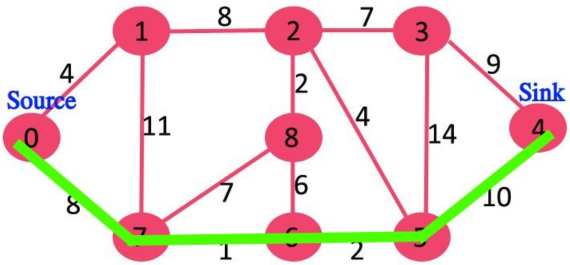

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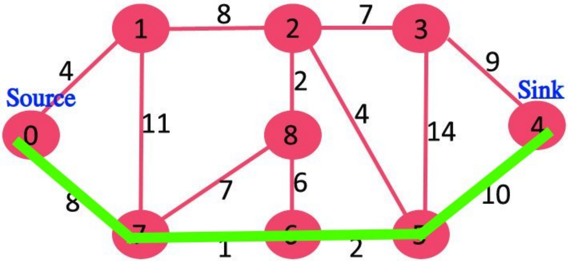

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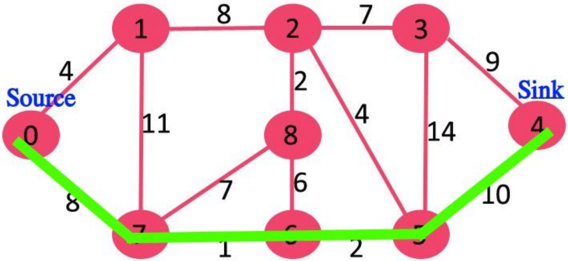

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What about parallel or distributed settings? Much harder!

What is the distributed setting?

- Undirected graph $G = (V, E)$ with $|V| = n$ nodes and **hop-diameter** D. "Network G" or "Network topology G" (read: undirected graph).
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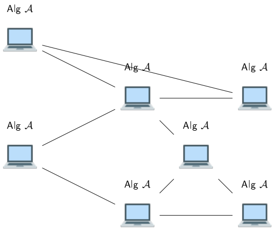

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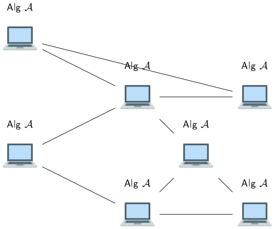

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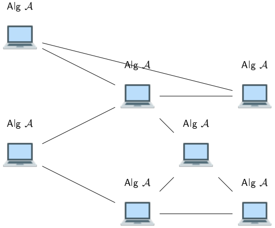

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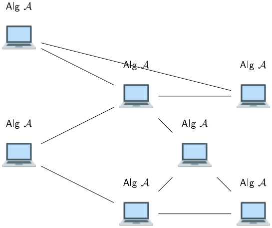

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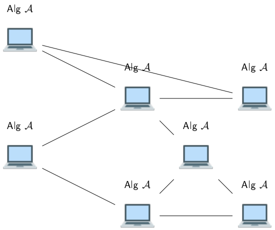

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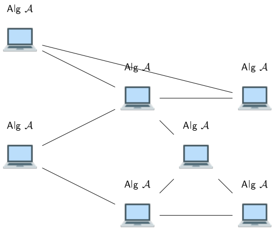

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Thm: $OPT(G) =_{\tilde{\Theta}(1)} ShortcutQuality(G)$ [Haeupler, Wajc, Zuzic; 2021]

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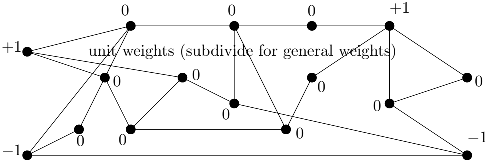

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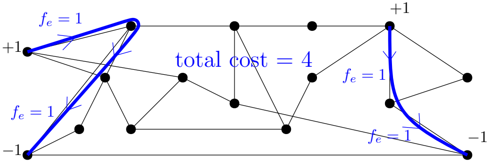

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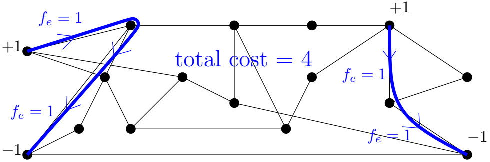

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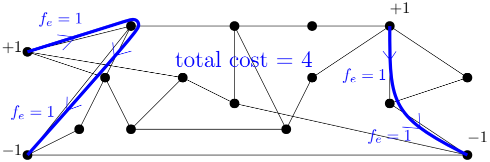

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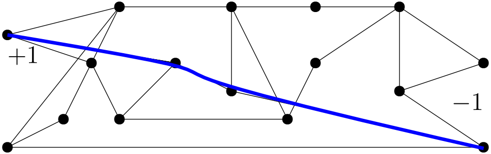
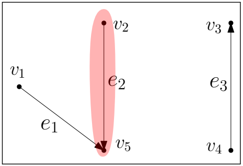
Note. Generalizes $(s-t)$ shortest path. (Also generalizes SSSP.)



Transshipment: a primal-dual formulation

Write the graph $G = (V, E)$ using the node-edge incidence matrix B.
Note: we orient edges arbitrarily.



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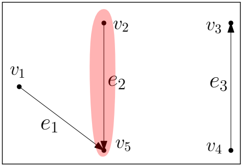

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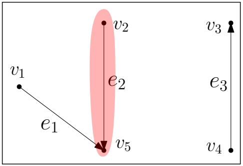

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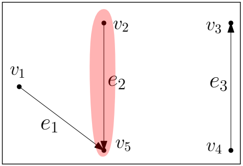

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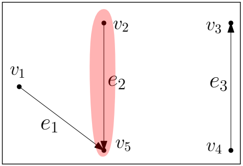

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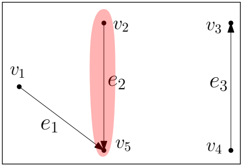

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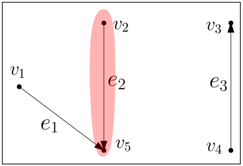

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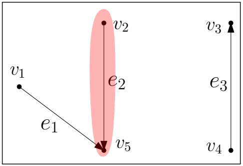

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$(B^T \phi)_e = \phi_a - \phi_b$ is height difference

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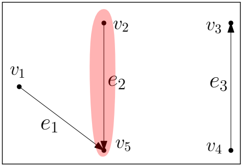

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{cccc} e_1 & \color{red}{e_2} & e_3 & \cdots \\ \hline +1 & & & \cdots \\ & \color{red}{+1} & & \cdots \\ & & -1 & \cdots \\ & & +1 & \cdots \\ -1 & \color{red}{-1} & & \cdots \end{array}$$

Primal.

$$\min_f \quad \|f\|_1 : Bf = d$$

$f_e = 0$ if no flow along e

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SP .. $f^* =$ shortest path from s to t

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$$\max_\phi \quad \langle d, \phi \rangle : \left\| B^\top \phi \right\|_\infty \leq 1.$$

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Transshipment: a primal-dual formulation

Write the graph $G = (V, E)$ using the node-edge incidence matrix B.
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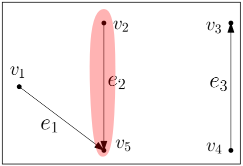

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Idea 2: Transshipment boosting (prior work)

Why transshipment? Isn't it harder?

Amazing property: we can boost a bad approximation to a good approximation.

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Theorem ([Sherman; 2013], [BFKL; 2016], [Zuzic; unpublished])

Fix G. Suppose we are given an oracle $O_G(\cdot)$ which, given a demand d, outputs an α-approximate feasible dual $O_G(d)$. There is an algorithm that produces a $(1+\varepsilon)$-approximate feasible dual by calling $O_G(\cdot)$ at most $\mathrm{poly}(\alpha, \varepsilon^{-1}, \log n)$ times.

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Corollary

Given such a (dual) $n^{o(1)}$-approximation oracle, we can solve $(1 + \frac{1}{n^{o(1)}})$-approximate transshipment in $n^{o(1)}$ oracle calls.

1. Introduction
2. Main Ideas
    - Idea 1: Transshipment generalizes shortest path (prior work)
    - Idea 2: Transshipment boosting (prior work)
    - Idea 3: Approximately Solving Transshipment (main contrib)
    - Idea 4: Distributed Implementation (contribution)
3. Conclusion

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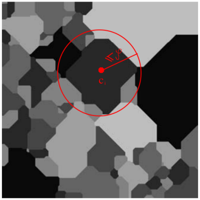

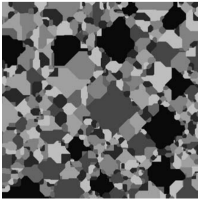

[Miller, Peng, Xu; 2013]

Definition

For a graph G, a low-diameter decomposition (LDD) of radius ρ is a distribution over node partitions called **clusters** $V = S_1 \sqcup \ldots \sqcup S_k$ along with centers $c_1 \in S_1, \ldots, c_k \in S_k$ such that:

1. For each i, the center c_i is within distance ρ of every other node in the induced subgraph $G[S_i]$, w.h.p.
2. For all $x, y \in V$, the probability x, y are in different clusters is at most $2^{\sqrt{\log n}} \cdot \frac{dist_G(u,v)}{\rho}$.

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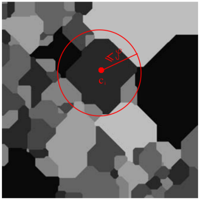
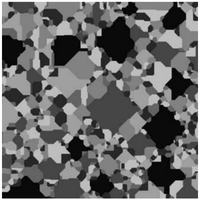
Theorem (Prior work [Haeupler, Li; 2018])

LDDs can be sampled in $OPT(G)n^{o(1)}$ CONGEST rounds.





[Miller, Peng, Xu; 2013]

Algorithm 0: Oblivious routing for TS.

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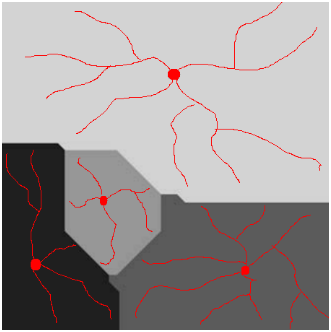

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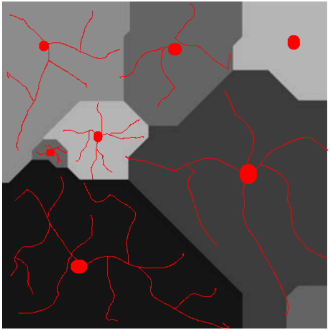
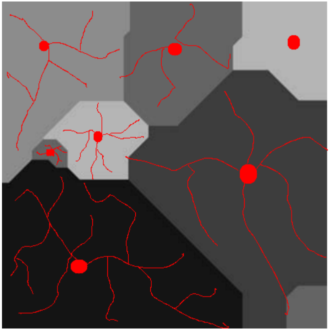


Algorithm 0: Oblivious routing for TS.

1. Let $\rho := 2^{(\log n)^{3/4}}$ (LDD radius).
2. For $i = 1, 2, \ldots, (\log n)^{1/4}$ repeat the following:
   1. For $j = 1, 2, \ldots, g := 2^{(\log n)^{3/4}}$.
      1. Sample an LDD with radius ρ^i.
      2. Each v sends $\frac{1}{g}$-fraction of its demend to the center of cluster containing v.
   2. Update the demand to reflect the transport.
3. Route all remaining demand to a common node along any spanning tree.



When $i = (\log n)^{1/4}$, radius is $\rho^i = \text{poly}(n)$ and LDD has a single cluster.

Analysis intuition.

Question: how does OPT change between steps?

Fix u, v at distance ℓ. Suppose at some step $d(u) = +1, d(v) = -1, d(\text{all else}) = 0$. Clearly, $\mathrm{OPT}_{before} = \ell$. Suppose we sampled an LDD of radius ρ. How does OPT change?

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In both cases, OPT increases by at most $2^{\sqrt{\log n}}$ factor. Hence after $(\log n)^{1/4}$ steps, it only increases by

$$\left(2^{\sqrt{\log n}}\right)^{(\log n)^{1/4}} = 2^{(\log n)^{3/4}} = n^{o(1)}$$

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Issue: On last slide, we proved the new OPT grows slowly .. BUT only looked at a fixed "pair" demand.

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After repeating LDD sampling $2^{(\log n)^{3/4}}$ times, we get concentration and it holds for all demands.

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Corollary

Transshipment can be solved in $n^{o(1)}$ Minor-Aggregation rounds. Hence it can be implemented in $OPT(G) \cdot n^{o(1)}$ CONGEST rounds.

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