Parallel Breadth-First Search and Exact Shortest Paths and Stronger Notions for Approximate Distances

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- Undirected graph with non-negative weights $w(e)$.
- Single source.

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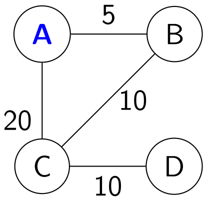

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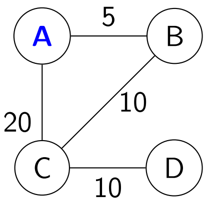

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We assume some source A is clear from context. We define:

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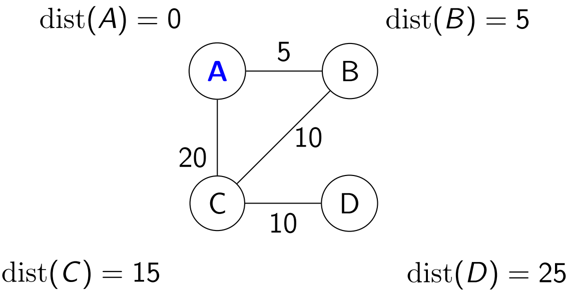

Definition

We assume some source A is clear from context. We define:

- **Exact distances**: $\mathrm{dist}(B)$. Sometimes $\mathrm{dist}(A, B)$.
- **Approximate distances:** $\tilde{d}(B)$.

Approximate distances.

Distance estimate = any function $\tilde{d} : V \to \mathbb{R}_{\geq 0}$.

Definition

A function $\tilde{d} : V \to \mathbb{R}_{\geq 0}$ is a **weak** $(1+\varepsilon)$**-approximation** (with respect to some source $s \in V$) if:

$$\forall v \in V \qquad dist(v) \leq \tilde{d}(v) \leq (1+\varepsilon)\mathrm{dist}(v).$$

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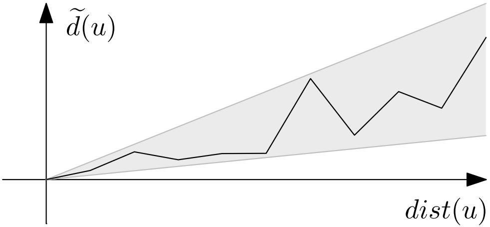
Example: G = a path graph (from left to right). Source = leftmost node.



weak approximate distances

Shortest path (i.e., distance computation) is an important building block. For example:

- Maximum flows [Edmonds-Karp'72] [Dinitz'70],
- Embeddings (embedding a graph into L1 [Bourgain'85]),
- Clustering (doing low-diameter decompositions [MPX'13]),
- Etc.

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Warning!

But these algorithms typically assume **exact** distance computations. This is easy in the sequential model (e.g., Dijkstra = simple and near linear).

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But in other models (parallel, distributed, streaming, etc.) exact distances are notoriously hard to compute. **Approximate** distances are much easier. But the above applications **typically fail** with (weak) approximations.

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2. **Main Result:** Find efficient algorithms to construct them from weak (usual) distance approximations.
3. **Consequence:** Give the first $\widehat{O}(m)$-work sublinear-depth parallel exact SSSP algorithm.
    - Not in this talk!
    - Our result: $\widehat{O}(m)$ work and $\widehat{O}(\sqrt{n})$ depth.
    - Same result achieved independently by [Cao, Fineman'23].

1. Introduction
2. New Stronger Notions of Distance Approximations.
   - Stronger Notion: Smothness
   - Stronger Notion: Tree-likeness
   - Smoothness + Tree-likeness = <3
3. Efficient Constructions: Lifting Weak Approximations to Smooth Ones
4. Conclusion

Stronger Notion: Smoothness

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Definition

A function (called "distance estimate") $\tilde{d} : V \to \mathbb{R}_{\geq 0}$ is a **smooth** $(1+\varepsilon)$**-approximation** (with respect to a source $s \in V$) if:

$$\tilde{d}(s) = 0 \text{ and}$$

$$\forall u, v \in V \qquad |\tilde{d}(u) - \tilde{d}(v)| \leq (1+\varepsilon)\mathrm{dist}(u, v).$$

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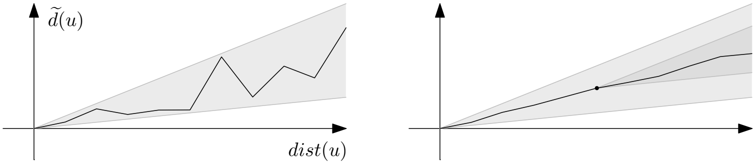

weak approximate distances smooth approximate distances

Stronger Notion: Tree-likeness

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Definition (Informal)

$\tilde{d} : V \to \mathbb{R}_{\geq 0}$ is **tree-like** if:

- $\tilde{d}(s) = 0$, and
- every other node v has a neighbor u whose estimate $\tilde{d}$ is smaller by at least $w(u, v)$.

(Check the full talk for formal details.)

Smoothness + Tree-likeness = <3

Theorem

The following two are equivalent:

- $\tilde{d}$ *is a smooth and tree-like* $(1+\varepsilon)$ *approximation (from source* s*).*

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- $\tilde{d}$ *is a smooth and tree-like* $(1+\varepsilon)$ *approximation (from source* s*).*
- *Given weights* $w : E \to \mathbb{R}_{\geq 0}$*, there exists a perturbation* $w'(e) \in [w(e), (1+\varepsilon)w(e)]$ *such that* $\tilde{d}(v) = \mathrm{dist}_{w'}(source = s, v)$*.*

Efficient Constructions: Lifting Weak Approximations to Smooth Ones

Theorem

Suppose we have an oracle that computes weak $(1 + \varepsilon)$-approximate distances.

There is an efficient algorithm that calls the oracle $O(\log n)$ times, asks for weak $(1 + \varepsilon/O(\log n))$-approximations on different graphs, and computes $(1 + \varepsilon)$-approximate smooth distances.

- Same for tree-likeness.
- I will only the main ideas behind efficiently turning weak $\rightarrow$ smooth.

Efficient Constructions: Lifting Weak Approximations to Smooth Ones

Goal: $\forall u, v \in V \qquad |\tilde{d}(u) - \tilde{d}(v)| \leq (1+\varepsilon)\mathrm{dist}(u, v).$

Definition

A function $\tilde{d}$ is (α, δ)-smooth if:

$$\forall u, v \in V \qquad |\tilde{d}(u) - \tilde{d}(v)| \leq (\alpha)\mathrm{dist}(u, v) + \delta.$$

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I will present an efficient algorithm that will transform $\tilde{d}$ from

$$(\alpha, \delta) \to (\alpha \cdot (1 + \frac{\varepsilon}{O(\log n)}), \delta/2).$$

Then we would be done! $(1, n^{100}) \to \ldots \to (1+\varepsilon, \frac{1}{n^{100}})$.

$(\alpha, \delta) \to (\alpha \cdot (1 + \frac{\varepsilon}{O(\log n)}), \delta/2)$

Algorithm Slow Partial Smoothing algorithm (n oracle calls)

Let G' be the graph G with distances multiplied by $(1 + \frac{\varepsilon}{2 \log n})\alpha$.
$\tilde{d} \leftarrow \mathrm{O}(G, \text{source} = s, \text{approx} = 1 + \frac{\varepsilon}{\log n})$
for each $u \in V(G)$ **do**
    $\tilde{d}_u \leftarrow \mathrm{O}(G', \text{source} = u, \text{approx} = \frac{\varepsilon}{10 \log n})$
    $\tilde{d}_u(\cdot) \leftarrow \tilde{d}(u) + \tilde{d}_u(\cdot)$
return $\tilde{d}_*(\cdot) = \min_{u \in V(G)} \tilde{d}_u(\cdot)$

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Intuition

When looking at nodes at most $\frac{\delta \log n}{\varepsilon}$ *close to* u*, they are already* $(\alpha \cdot (1 + \frac{\varepsilon}{O(\log n)}), \delta/2)$*-smooth.*

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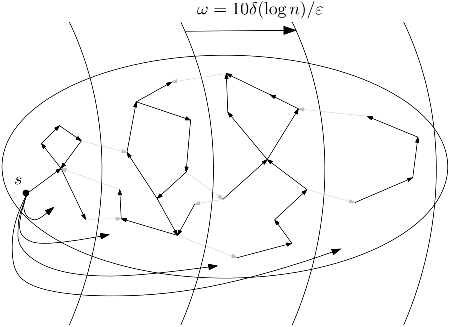
When $\operatorname{dist}(u, v) > \frac{\delta \log n}{\varepsilon}$, then $\tilde{d}_u(v) > \tilde{d}(v)$.

Q: How to reduce the number of oracle calls from n to $O(1)$?

A: (Carefully) carve out the graph into strips of width $\omega := \frac{10\delta \log n}{\varepsilon}$. Connect all nodes to the source. Call the oracle. (And fix certain kind of mistakes.)



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   - Iterative Smoothing
   - $(\alpha, \delta) \to (\alpha \cdot (1 + \frac{\varepsilon}{O(\log n)}), \delta/2)$
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Thank you!