Universally-Optimal Distributed Algorithms for Known Topologies

Speaker: Goran Zuzic

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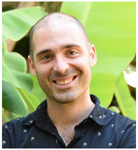
Bernhard Haeupler
CMU / ETH Zürich


David Wajc
Stanford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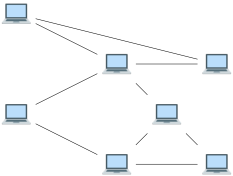
Introduction

Setting: we are given some specific distributed network G.



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- Processors in a supercomputer.
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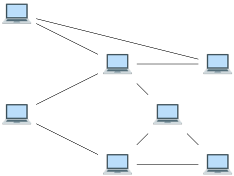

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- SSSP (single-source shortest path),
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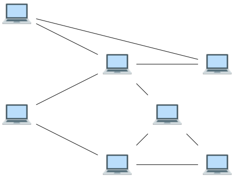

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Goal

Design a distributed MST protocol that is as fast as possible on G.

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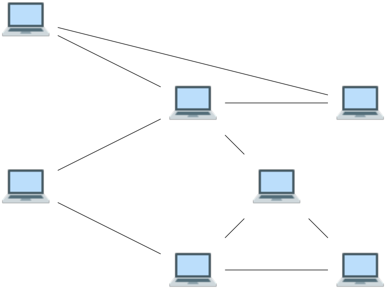
Most prior work focuses only on pathological worst-case graphs G.

Our question: what is the optimal running time for non-worst-case networks G.

Model

CONGEST model



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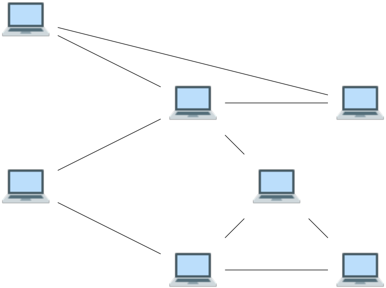

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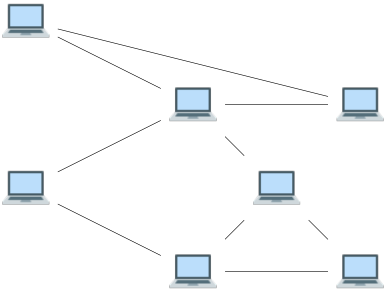

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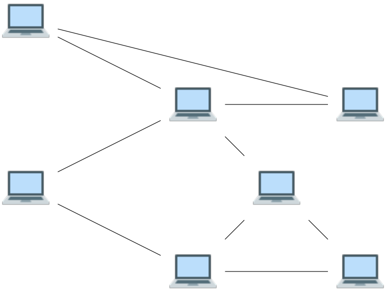

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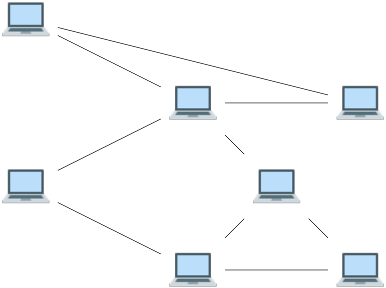

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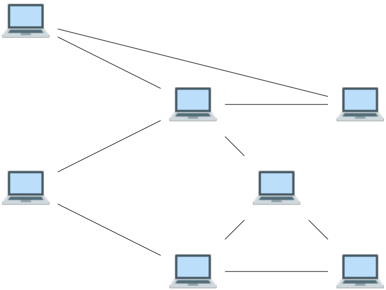

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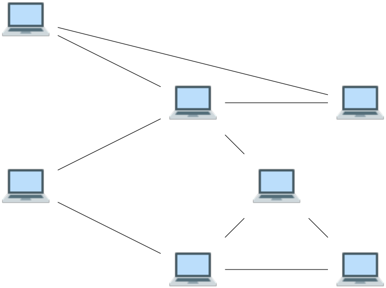

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- Objective: minimize # rounds.

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We answer both of these questions.

Our results

For every undirected graph G there is a graph parameter **ShortcutQuality**(G) [Ghaffari, Haeupler, 2015].

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- Lower bound (impossibility view):

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Distributed MST requires at least $\tilde{\Omega}(\textbf{ShortcutQuality}(G))$ time.

- Upper bound (algorithmic view):

Theorem

Distributed MST can be solved in $\tilde{O}(\textbf{ShortcutQuality}(G))$ time if the topology is known in advance (but not the input!).

Consequences 1/2

- New result: universal optimality = as fast as possible on every network.
  - Intuition: “perfectly adapts to the network!”
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  - Intuition: "perfectly adapts to the network!"
  - We achieve it in the known-topology setting!
- Old notion: "existential optimality" = optimal in a class of graphs.
  - Depends on the parameterization ($\tilde{O}(\sqrt{n} + D)$ is optimal only when parameterizing via n and D).
  - Universal optimality is optimal with respect to all parameterizations.

- Same method works for many other problems:
  - (Approx) distributed SSSP.
  - (Approx) distributed mincut.
  - Distributed connectivity verification.
- Moreover, $\tilde{O}(\textbf{ShortcutQuality}(G))$ characterizes all of them.

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- Moreover, $\tilde{O}(\textbf{ShortcutQuality}(G))$ characterizes all of them.
- These problems inter-reduce to each other.

Shortcut definition

Definition ([Ghaffari, Haeupler, 2015])

In a graph $G = (V, E)$ we are given connected node-disjoint parts $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$, $P_i \subseteq V$. A shortcut of quality Q for $\mathcal{P}$ is:

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Definition

$$\textbf{ShortcutQuality}(G) = \max_{\mathcal{P}} \min_{\text{shortcut for } \mathcal{P}} \mathrm{quality}(\mathcal{P})$$

Shortcut application: part-wise aggregation

Example (Part-wise aggregation [Ghaffari, Haeupler, 2015])

We are given connected node-disjoint parts $\{P_1, P_2, \ldots, P_k\}$. Each node v has a $O(\log n)$-bit private input x_v. Each part needs to learn the minimum of the inputs in it.

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Lemma ([Ghaffari, Haeupler, 2015])

Given a quality-Q shortcut on $\{P_1, \ldots, P_k\}$, we can solve the part-wise aggregation problem in $\tilde{O}(Q)$ rounds.

Hints on solving part-wise aggregation via quality-Q shortcuts:

- Assume each part P_i has a leader $v_i \in P_i$ (easy exercise).
- All parts concurrently build a BFS tree of $H_i := G[P_i] + G[F_i]$:
  - The leader v_i becomes "active" in a uniformly random time $\{0, \ldots, Q\}$.
  - When a node becomes active, it spreads a message along its neighbors in H_i (only once).
  - A node becomes active the first time it hears a message from part i.
  - Analysis: in every round at most $O(\log n)$ messages are scheduled to pass through an edge, with high probability. We send those messages by subdividing each round into $O(\log n)$ subrounds. Since the BFS tree has depth Q, the process completes in $O(Q \log n)$ subdivided rounds.
- Spread the maximum using the BFS tree using the same idea (randomly delay each part by $\{0, 1, \ldots, Q\}$ and flood-fill the tree).

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- Repeat $O(\log n)$ times.

1. Introduction

2. Preliminary: Shortcuts

3. High-level technical overview
   - Upper bound: shortcut construction
   - Lower bound: distributed disjointness task
   - Lower bound: ingredients

4. Lower bound: more details

5. Conclusion and Open Questions

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Theorem (Upper bound)

Suppose all nodes know the topology G upfront. We can construct shortcuts of near-optimal quality Q in $\tilde{O}(Q)$ rounds.

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Theorem

The distributed disjointness task on each subset of paths $\{P_1, \ldots, P_k\}$ can be solved in Q rounds.

if and only if

There exists shortcut of quality $\tilde{O}(Q)$ for $P_1, \ldots, P_k$.

- See "Network Coding Gaps for Completion Times of Multiple Unicasts" [HWZ FOCS'20] on Youtube.

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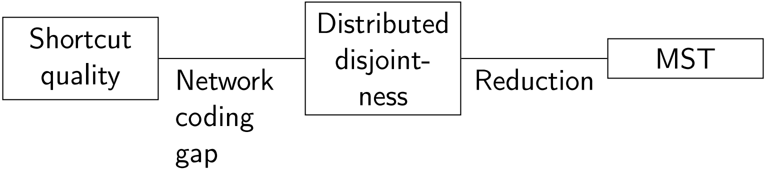

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Equivalent:

Lemma

Given any set of node-disjoint paths $P_1, \ldots, P_k$ we can construct shortcuts of quality $\tilde{O}(T_{MST})$ on them.

Hints on why is it sufficient to consider only node-disjoint paths P_i (instead of arbitrary connected and node-disjoint subsets):

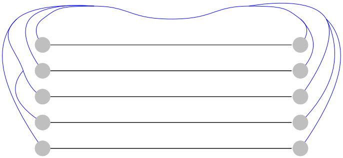
- Let $T_1, \ldots, T_k$ be some spanning trees of $P_1, \ldots, P_k$ (note: P_i is connected).
- Root each T_i and consider the heavy-light decomposition of T_i, which decomposes any tree into a number of paths such that for any path p there are at most HL-depth(p) $\leq O(\log n)$ paths on the root-to-p path in T_i.
- For $i = O(\log n)$ down to 1 do:
  - Consider all paths of HL-depth(p) $= i$.
  - By assumption, we can construct shortcuts of quality $\tilde{O}(Q)$ on them (each path is its own part). Construct it.
- The shortcut of P_i is the union of the shortcuts associated paths of the heavy-light decomposition of T_i.
- Since the shortcuts of P_i's were constructed by $O(\log n)$ calls to the path-wise shortcut oracle, their quality increases by a negligible $O(\log n)$ compared to the shortcuts of the paths.

Disjointness gadget: definition

Definition

A disjointness gadget of a set of node-disjoint paths $P_1, \ldots, P_k$ is a connected subset of edges F that touches the heads/tails of each path, but does not otherwise intersect the interior.[1]

[1] We also allow $O(1)$ "exception intervals" of length $O(D)$ on each P_i where F can intersect.

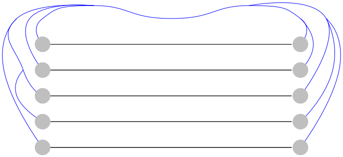


A disjointness gadget

Observation

Let F be a disjointness gadget of node-disjoint paths $\mathcal{P}$. Using a single call to the MST oracle, we can solve the distributed disjointness task on $\mathcal{P}$.

Idea: Given Alice/Bob inputs x, y we assign MST costs such that MST has cost 0 if and only if x and y are disjoint.



Theorem (Main technical contribution of the paper)

Given any set of node-disjoint paths P, there exists a disjointness gadget on a subset $P' \subseteq P$ of size $|P'| \geq \frac{1}{\text{poly}(\log n)}|P|$.

Completing the proof:

- It is sufficient to construct of quality T_{MST} on arbitrary node-disjoint paths $\mathcal{P}$.

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- Via network coding gap, there exists shortcut on $\mathcal{P}'$ of quality $\tilde{O}(T_{MST})$.
- Remove $\mathcal{P}'$ from $\mathcal{P}$ and repeat $\tilde{O}(1)$ times. The final shortcut is still of quality $\tilde{O}(T_{MST})$.

Disjointness gadget: construction

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Theorem (Simplified construction)

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  - p_i "deletes" all paths it encounters.
  - Self-intersecting parts are exceptional intervals.
- Each p_i deletes $O(D)$ other paths.
- So, there must exist an independent subset $|P'| \geq \frac{1}{O(D)}|P|$.

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Thank you!